# MATH 232: Final Exam 2021

**Exercise 1.** There are 10 blue cards numbered from 1 to 10 and 5 green cards numbered from 1 to 5. How many different permutations of the cards can you obtain,

**(a)** if your eye can distinguish the colors and can see the numbers?

**(b)** if your eye can distinguish the colors but cannot see the numbers?

**(c)** if your eye cannot distinguish the colors but can see the numbers?

**Exercise 2.** Assume that a person gets the C-virus when exposed to it with probability
(i) 0.01 if the person is vaccinated and wearing a mask;
(ii) 0.05 if the person is vaccinated and not wearing a mask;
(iii) 0.1 if the person is not vaccinated and wearing a mask;
(iv) 0.5 if the person is not vaccinated and not wearing a mask.
Assume that a person wears a mask with probability 1/2 and that a person is vaccinated with probability 1/3. Furthermore, assume that people wear masks independently of being vaccinated. What is the probability that a person was vaccinated, given that he/she wore a mask and got the C-virus when exposed to it?

**Exercise 3.** Let $\theta$ be a random variable that is uniformly distributed in the interval $[-\pi, \pi]$ (this means that the density function of $\theta$ equals $\frac{1}{2\pi}$ on the interval $[-\pi, \pi]$ and 0 otherwise). Let $X = \cos(\theta)$ and $Y = \sin(\theta)$.

**(a)** Are $X$ and $Y$ uncorrelated?

**(b)** Are $X$ and $Y$ independent?

**Exercise 4.** Assume that in a class of $n$ students, any combination of 3 students forms a friend-triplet (i.e., 3 students who are all friends) with probability $e^{-n}$. Show that the number of friend-triplets in a class of $n$ students is greater or equal to 1 with vanishing probability as $n$ tends to infinity.

*Hint:* you can use Markov's inequality to approach this problem.

**Exercise 5.** Let $X_1, X_2$ be i.i.d. $\mathcal{N}(0,1)$ and let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

**(a)** What is the joint distribution of $(Y_1, Y_2)$?

**(b)** Are $Y_1$ and $Y_2$ independent?

**Exercise 6.** Let $Y_1, \ldots, Y_n$ be i.i.d. such that $Y_i$ is uniformly distributed on $[a, b]$, where $a < b$, (this means that the density function of $Y_i$ equals $\frac{1}{b-a}$ on the interval $[a, b]$ and 0 otherwise). Find estimators for $a$ and $b$ using the method of moments.

*Hint:* you may use the fact that $(x^3 - y^3) = (x - y)(x^2 + xy + y^2)$.

**Exercise 7.** Let $Y_1, \ldots, Y_n$ be i.i.d. such that $Y_i$ is uniformly distributed on $[0, \theta]$.

**(a)** Show that $\max(Y_1, \dots, Y_n)/\theta$ is a pivot.

**(b)** Using this pivot, give an equi-tailed two-sided confidence interval for $\theta$ with confidence level $1 - \alpha = 7/8$ (i.e. $\alpha = 1/8$) in the case where $n = 2$ and $\max(Y_1, Y_2) = 1$.

**Exercise 8.** Consider the binary hypothesis test where a single random variable $Y$ is observed. Under $H_0$, the random variable $Y$ is uniformly distributed on $[0, 3]$, and under $H_1$, the random variable $Y$ is uniformly distributed on $[1, 4]$.

**(a)** Give a test (i.e., define the set of values of $Y$ for which you declare $H_0$ or $H_1$) such that the sum of the false negative and false positive probabilities is minimized.

**(b)** Give a test (i.e., define the set of values of $Y$ for which you declare $H_0$ or $H_1$) such that either the false negative or the false positive probability is zero and the other of the two probabilities is minimized.